A talk given at Capital Normal University (April 15, 2011)
and Conference on Number Theory and Related Fields
in honor of Prof. Keqin Feng (Hefei and Huangshan, June 7-12)

Conjectures and Results on x^2 mod p^2 with $4p = x^2 + dy^2$

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Two Typical Conjectures in this Talk

Conjecture (Z. W. Sun, Nov. 13, 2009). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3 \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1 \ \&\ p=x^2+7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1. \end{cases}$$

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Conjecture (Z. W. Sun, Jan. 2, 2011). We have

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-256)^k}\binom{2k}{k}^2 a_k = \frac{24}{\pi},$$

where a_k is the coefficient of x^k in $(x^2+x+16)^k$.

`Mathematica` Program:

```
T[n_]:=If[n>0,Coefficient[(x^2+x+16)^n,x^n],1]
S[n_]:=Sum[(30k+7)Binomial[2k,k]^2*T[k]/(-256)^k,{k,0,n}]
Print[N[S[200]Pi,20]]
```

Output: `24.000000000000000000`

Part A. Series for $1/\pi$

Gaussian hypergeometric series

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Note that $(1)_n = n!$.

Classical Gaussian hypergeometric series:

$${}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n} \cdot \frac{x^n}{n!},$$

where $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r < 1$, $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r < 1$, and $|x| < 1$.

Gaussian hypergeometric series

$$y = {}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right)$$

satisfies the differential equation:

$$\left(\theta \prod_{t=1}^{r}(\theta + \beta_t - 1) - x \prod_{s=0}^{r}(\theta + \alpha_s)\right) y = 0$$

where

$$\theta = x\frac{d}{dx}.$$

Clausen's Identity:

$$\begin{aligned} &{}_2F_1\left(2a, 2b; a + b + 1/2 \mid x\right)^2 \\ =&{}_3F_2\left(2a, 2b, a + b; a + b + 1/2, 2a + 2b \mid 4x(1 - x)\right). \end{aligned}$$

In the case $a = b = 1/4$, it gives the identity

$$\left(\sum_{k=0}^{\infty} \binom{2k}{k}^2 x^k\right)^2 = \sum_{k=0}^{\infty} \binom{2k}{k}^3 (x(1 - 16x))^k.$$

Series for $1/\pi$

G. Bauer (1859):

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k} = \frac{2}{\pi}.$$

In his famous letter to Hardy, S. Ramanujan mentioned the above series as one of his discoveries.

In 1914 S. Ramanujan published his first paper in England *Modular equations and approximations to* π, Quart. J. Math. (Oxford), 45(1914), 350–372.

Towards the end of this paper, he wrote "*I shall conclude this paper by giving a few series for* $1/\pi$". Then he listed 17 series for $1/\pi$ and briefly mentioned that the first three series are related to the classical theory of elliptic functions.

Elliptic integrals

Complete elliptic integrals (with $0 < k < 1$):

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad \text{(the first kind)},$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta \quad \text{(the seond kind)}.$$

Legendre's Relation:

$$E(k)K(\sqrt{1-k^2}) + E(\sqrt{1-k^2})K(k) - K(k)K(\sqrt{1-k^2}) = \frac{\pi}{2}.$$

A Central Result:

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \;\middle|\; k^2\right) = \frac{2}{\pi} K(k) = \varphi^2(q)$$

where $q = e^{-\pi K(\sqrt{1-k^2})/K(k)}$ and

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{(theta function)}.$$

Series for $1/\pi$ given by Ramanujan

Two of the 17 series for $1/\pi$ recorded by Ramanujan:

$$\sum_{k=0}^{\infty} \frac{6k+1}{4^k} \cdot \frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty} (6k+1) \frac{\binom{2k}{k}^3}{256^k} = \frac{4}{\pi},$$

(proved by S. Chowla in 1928)

$$\sum_{k=0}^{\infty} \frac{26390k+1103}{99^{4k}} \cdot \frac{(1/2)_k (1/4)_k (3/4)_k}{(1)_k^3}$$

$$= \sum_{k=0}^{\infty} \frac{26390k+1103}{396^{4k}} \binom{4k}{k,k,k,k} = \frac{99^2}{2\pi\sqrt{2}}.$$

In 1985 Jr. R. W. Gosper used the last series of Ramanujan to calculate $17,526,100$ digits of π (a world record at that time).

In 1987 Jonathan Borwein and Peter Borwein succeeded in proving all the 17 Ramanujan series for $1/\pi$.

Ramanujan-type series for $1/\pi$

General forms:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k}, \quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}, \quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

There are totally 36 known Ramanujan-type series for $1/\pi$ with a, b, m rational.

D. V. Chudnovsky and G. V. Chudnovsky (1987):

$$\sum_{k=0}^{\infty}\frac{545140134k+13591409}{(-640320)^{3k}}\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k} = \frac{3\times 53360^2}{2\pi\sqrt{10005}}.$$

Remark. This yielded the record for the calculation of π during 1989-1994.

Other known series for $1/\pi$

T. Sato (2002, announced):

$$\sum_{k=0}^{\infty}(20n+10-3\sqrt{5})\left(\frac{\sqrt{5}-1}{2}\right)^{12n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\frac{20\sqrt{3}+9\sqrt{15}}{6\pi}.$$

Yifan Yang (2005, unpublished):

$$\sum_{n=0}^{\infty}\frac{4n+1}{36^n}\sum_{k=0}^{n}\binom{n}{k}^4=\frac{18}{\sqrt{15}\,\pi}.$$

H. H. Chan, S. H. Chan and Z. G. Liu (2004, Adv. in Math.)

$$\sum_{n=0}^{\infty}\frac{5n+1}{64^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}=\frac{8}{\sqrt{3}\,\pi}.$$

H. H. Chan and H. Verill (2009, Math. Res. Lett.), M. D. Rogers (2009, Ramanujan J.)

$$\sum_{n=0}^{\infty}\frac{3n+1}{(-32)^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}=\frac{2}{\pi}.$$

Connections to modular forms

Many known series for $1/\pi$ have the form

$$\sum_{k=0}^{\infty} \frac{bk+c}{m^k} \binom{2k}{k} u_k = \frac{C}{\pi},$$

where $u_{-1} = 0,\ u_0 = 1$ and

$$(k+1)^2 u_{k+1} = (Ak^2 + Ak + B) u_k + Ck^2 u_{k-1} \ (k = 1, 2, 3, \ldots),$$

and there are modular functions (i.e., meromorphic modular forms of weight 0) $x(\tau)$ and $\tilde{x}(\tau)$ such that

$$F(\tau) = \sum_{k=0}^{\infty} u_k (x(\tau))^k \quad \text{and} \quad \tilde{F}(\tau) = \sum_{k=0}^{\infty} \binom{2k}{k} u_k (\tilde{x}(\tau))^k$$

are modular forms of weights 1 and 2 respectively.

Zagier's contribution

Don Zagier (2009) investigated what integer sequence $\{u_n\}$ satisfies $u_{-1} = 0,\ u_0 = 1$, and the Apéry-like recurrence relation

$$(k+1)^2 u_{k+1} = (Ak^2 + Ak + B)u_k + Ck^2 u_{k-1} \ (k = 1, 2, 3, \ldots).$$

For example, he noted that if $(A, B, C) = (7, 2, 8)$, then $u_n = \sum_{k=0}^{n} \binom{n}{k}^3$ and

$$\frac{\eta_2 \eta_3^6}{\eta_1^2 \eta_6^3} = \sum_{n=0}^{\infty} u_n \left(\frac{\eta_1^3 \eta_6^9}{\eta_2^3 \eta_3^9} \right)^n,$$

where

$$\eta_m := q^{m/24} \prod_{n=1}^{\infty} (1 - q^{mn}) = \eta(m\tau)$$

with $q = e^{2\pi i \tau}$ and $\mathrm{Im}(\tau) > 0$.

Connections to differential equations

Let $u_{-1} = 0,\ u_0 = 1$ and

$$(k+1)^2 u_{k+1} = (Ak^2 + Ak + B)u_k + Ck^2 u_{k-1}\ (k = 1, 2, 3, \ldots).$$

Then

$$y = \sum_{k=0}^{\infty} \binom{2k}{k} u_k x^k$$

satisfies the third-order differential equation

$$x^2(1 - 4Ax - 16Cx^2)y''' + 3x(1 - 6Ax - 32Cx^2)y''$$
$$+ (1 - (12A + 4B)x - 108Cx^2)y' - 2(b + 6Cx)y = 0.$$

For $f(x) = \sum_{k=0}^{\infty} u_k x^k$ and $\tilde{f}(x) = \sum_{k=0}^{\infty} \binom{2k}{k} u_k x^k$, there is a Clausen-type relation

$$(1 + Cx^2)f(x)^2 = \tilde{f}\left(\frac{x(1 - Ax - Cx^2)}{(1 + Cx^2)^2}\right)$$

which can be verified via `Maple` or `Mathematica`.

Comments from Baruah, Berndt and Chan

S. Ramanujan attributed his mathematical discoveries to inspirations from the God. He once said: "*An equation for me has no meaning, unless it represents a thought of God.*"

At the end of the article *Ramanujan's series for $1/\pi$: a survey* [Amer. Math. Monthly 116(2009)] by N. D. Baruah, B. C. Berndt and H. H. Chan, the authors wrote the following comments:

One test of "good" mathematics is that it should generate more "good" mathematics. Readers have undoubtedly concluded that Ramanujan's original series for $1/\pi$ have shown the seeds for an abundant crop of "good" mathematics.

Generalized central trinomial coefficients

For real numbers b and c, we define

$$T_n(b,c) := [x^n](x^2+bx+c)^n$$

(the coefficient of x^n in $(x^2+bx+c)^n$)

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k.$$

Recursion: $T_0(b,c)=1$, $T_1(b,c)=b$, and

$$(n+1)T_{n+1}(b,c) = (2n+1)bT_n(b,c) - ndT_{n-1}(b,c)\ (n>0),$$

where $d=b^2-4c$. It is known that if $d \neq 0$ then

$$T_n(b,c) = \sqrt{d}^n P_n\left(\frac{b}{\sqrt{d}}\right)$$

where

$$P_n(x) := \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k$$

is the Legendre polynomial of degree n.

Asymptotic Behavior of $T_n(b,c)$

By the Laplace-Heine formula, for $x \notin [-1,1]$ we have

$$P_n(x) \sim \frac{(x+\sqrt{x^2-1})^{n+1/2}}{\sqrt{2n\pi}\sqrt[4]{x^2-1}} \quad \text{as } n \to +\infty.$$

It follows that if $b>0$ and $c>0$ then

$$T_n(b,c) \sim f_n(b,c) := \frac{(b+2\sqrt{c})^{n+1/2}}{2\sqrt[4]{c}\sqrt{n\pi}}.$$

as $n \to +\infty$. Note that $T_n(-b,c) = (-1)^n T_n(b,c)$.

Conjecture (Sun, 2011): For $b>0$ and $c>0$, we have

$$T_n(b,c) = f_n(b,c)\left(1+\frac{b-4\sqrt{c}}{16n\sqrt{c}}+O\left(\frac{1}{n^2}\right)\right)$$

as $n \to +\infty$. If $c>0$ and $b=4\sqrt{c}$, then

$$\frac{T_n(b,c)}{\sqrt{c}^n} = \frac{3\times 6^n}{\sqrt{6n\pi}}\left(1+\frac{1}{8n^2}+\frac{15}{64n^3}+\frac{21}{32n^4}+O\left(\frac{1}{n^5}\right)\right).$$

If $c<0$ and $b \in \mathbb{R}$ then

$$\lim_{n\to\infty} \sqrt[n]{|T_n(b,c)|} = \sqrt{b^2-4c}.$$

My conjectural series involving $T_k(b,c)$ for $1/\pi$

In Jan.-Feb. 2011, I introduced 40 series for $1/\pi$ of the following new types with a,b,c,d,m integers and $mbcd(b^2-4c)$ nonzero.

Type I. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_k(b,c)/m^k$.

Type II. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_k(b,c)/m^k$.

Type III. $\sum_{k=0}^{\infty}(a+dk)\binom{4k}{2k}\binom{2k}{k} T_k(b,c)/m^k$.

Type IV. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_{2k}(b,c)/m^k$.

Type V. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_{3k}(b,c)/m^k$.

Recall that a series $\sum_{k=0}^{\infty} a_k$ is said to *converge at a geometric rate with ratio r* if

$$\lim_{k\to+\infty}\frac{a_{k+1}}{a_k}=r\in(0,1).$$

My conjectural series of type I

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-256)^k}\binom{2k}{k}^2T_k(1,16)=\frac{24}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-1024)^k}\binom{2k}{k}^2T_k(34,1)=\frac{12}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{30k-1}{4096^k}\binom{2k}{k}^2T_k(194,1)=\frac{80}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{42k+5}{4096^k}\binom{2k}{k}^2T_k(62,1)=\frac{16\sqrt{3}}{\pi}.$$

Remark. The first identity was found by me soon after I waked up in the deep night of Jan. 1, 2011. This began my discovery of many new series for $1/\pi$.

My conjectural series of type II

I have 12 conjectural series of type II. Here are five of them.

$$\sum_{k=0}^{\infty}\frac{15k+2}{972^k}\binom{2k}{k}\binom{3k}{k}T_k(18,6)=\frac{45\sqrt{3}}{4\pi},$$

$$\sum_{k=0}^{\infty}\frac{91k+12}{10^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(10,1)=\frac{75\sqrt{3}}{2\pi},$$

$$\sum_{k=0}^{\infty}\frac{6930k+559}{102^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(102,1)=\frac{1445\sqrt{6}}{2\pi},$$

$$\sum_{k=0}^{\infty}\frac{210k-7157}{198^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(287298,1)=\frac{114345\sqrt{3}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{63k+11}{(-13500)^k}\binom{2k}{k}\binom{3k}{k}T_k(40,1458)=\frac{25}{12\pi}(3\sqrt{3}+4\sqrt{6}).$$

Remark. The 4th series converges very slow (with geometric ratio 71825/71874), even 2000 terms could not contribute one digit. Prof. G. Almkvist wondered how I could find the identity.

My conjectural series of type III

$$\sum_{k=0}^{\infty}\frac{85k+2}{66^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(52,1)=\frac{33\sqrt{33}}{\pi},$$
$$\sum_{k=0}^{\infty}\frac{28k+5}{(-96^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(110,1)=\frac{3\sqrt{6}}{\pi},$$
$$\sum_{k=0}^{\infty}\frac{40k+3}{112^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(98,1)=\frac{70\sqrt{21}}{9\pi},$$
$$\sum_{k=0}^{\infty}\frac{80k+9}{264^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(257,256)=\frac{11\sqrt{66}}{2\pi},$$
$$\sum_{k=0}^{\infty}\frac{80k+13}{(-168^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)=\frac{14\sqrt{210}+21\sqrt{42}}{8\pi}.$$

Remark. Some mathematicians (including my twin brother Z. H. Sun) wondered how I could find the last identity involving $14\sqrt{210}+21\sqrt{42}$.

My conjectural series of type IV

I have 18 conjectural series of type IV. Here are five of them.

$$\sum_{k=0}^{\infty} \frac{340k+59}{(-480^2)^k} \binom{2k}{k}^2 T_{2k}(62,1) = \frac{120}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{13940k+1559}{(-5760^2)^k} \binom{2k}{k}^2 T_{2k}(322,1) = \frac{4320}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{14280k+899}{39200^{2k}} \binom{2k}{k}^2 T_{2k}(198,1) = \frac{1155\sqrt{6}}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{57720k+3967}{439280^{2k}} \binom{2k}{k}^2 T_{2k}(5778,1) = \frac{2890\sqrt{19}}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{1615k-314}{243360^{2k}} \binom{2k}{k}^2 T_{2k}(54758,1) = \frac{1989\sqrt{95}}{4\pi}.$$

Remark. I conjectured that my list of the 18 series of type IV is complete! Prof. G. Almkvist asked me why I thought so.

My conjectural series of type V

$$\sum_{k=0}^{\infty}\frac{1638k+277}{(-240)^{3k}}\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)=\frac{44\sqrt{105}}{\pi}.$$

$u_k=\binom{3k}{k}T_{3k}(62,1)$ satisfies a very complicated recursion:

$$\begin{aligned}&(n+2)^2(2n+1)(2n+3)(8652n^2+11536n+3525)u_{n+2}\\=&372(2n+1)(6n+7)(25021584n^4+116767392n^3\\&+188134216n^2+121113048n+25958565)u_{n+1}\\&-127401984000(3n+1)^2(3n+2)^2(8652n^2+28840n+23713)u_n\\&-9(n+2)P(n)62^n\binom{2n+2}{n}\binom{3n+2}{n}\binom{3n+2}{2n},\end{aligned}$$

where

$$\begin{aligned}P(n):=&31420906020n^5+136307337012n^4+127456779135n^3\\&-126369328953n^2-174985958380n+705000.\end{aligned}$$

Comments

Concerning my 40 conjectural series involving $T_k(b,c)$, I have contacted many experts related to π-series such as *G. Almkvist, G. Andrews, B. C. Berndt, H. H. Chan, S. Cooper, Y. Yang, D. Zagier*, and they felt that those conjectures could not be proved by the current tools used to establish Ramanujan-type series.

In a message to me, Prof. S. Cooper made the following comments: "**They are completely mysterious!**"

Besides the 40 series involving $T_k(b,c)$, I have totally 124 conjectural series for powers of π and other constants, 118 of which are for $1/\pi$. For the full list, see my article *List of conjectural series for powers of π and other constants* `http://arxiv.org/abs/1102.5649`

All my conjectural series come from **combinations of philosophy, intuition, inspiration, experience and computation!**

More examples of my conjectural series

$$\sum_{n=0}^{\infty}\frac{1054n+233}{480^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}(-1)^k8^{2k-n}=\frac{520}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{162n+17}{320^n}\binom{2n}{n}\sum_{k=0}^{n}(-1)^k\binom{-1/4}{k}^2\binom{-3/4}{n-k}=\frac{16\sqrt{10}}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{1500000n+87659}{(-1000004)^n}\binom{2n}{n}\sum_{k=0}^{n}(-1)^k\binom{-1/3}{k}^2\binom{-2/3}{n-k}=\frac{16854\sqrt{267}}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{18n^2+7n+1}{(-128)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{-1/4}{k}^2\binom{-3/4}{n-k}^2=\frac{4\sqrt{2}}{\pi^2},$$

$$\sum_{n=0}^{\infty}\frac{40n^2+26n+5}{(-256)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}=\frac{24}{\pi^2}.$$

Remark. I think nobody could understand how I found the 3rd series which converges very fast (20 terms contribute 100 digits).

Part B. On x^2 mod p^2 with $4p = x^2 + dy^2$

Gauss' congruence

Gauss' Congruence. Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p.$$

Further Refinement of Gauss' Result (Chowla, Dwork and Evans, 1986):

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

It follows that

$$\binom{(p-1)/2}{(p-1)/4}^2 \equiv 2^{p-1}(4x^2 - 2p) \pmod{p^2}.$$

Determine x in $p = x^2 + 7y^2$

It is well known that the only imaginary quadratic fields with class number one are those $\mathbb{Q}(\sqrt{-d})$ with

$$d = 1, 2, 3, 7, 11, 19, 43, 67, 163.$$

In 1977, A. R. Rajwade proved that for any odd prime p we have

$$\sum_{x=0}^{p-1}\left(\frac{x^3+21x^2+112x}{p}\right)$$
$$=\begin{cases} -2x(\frac{x}{7}) & \text{if } (\frac{p}{7})=1 \ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0 & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6 \pmod 7).\end{cases}$$

Determine x in $4p = x^2 + dy^2$ with $d = 11, 19, 43, 67, 163$

Let p be an odd prime.

Via elliptic curves with complex multiplication, it is known [cf. Acta Arith. 40(1982) and JNT 19(1984), 55(1995), 61(1996)] that for $d = 11, 19, 43, 67, 163$ we have

$$\sum_{x=0}^{p-1} \left(\frac{f_d(x)}{p}\right) = \begin{cases} (\frac{2}{p})(\frac{x}{d})d & \text{if } (\frac{p}{d}) = 1 \ \&\ 4p = x^2 + dy^2 \ (x, y \in \mathbb{Z}), \\ 0 & \text{if } (\frac{p}{d}) = -1, \end{cases}$$

where

$$\begin{aligned} f_{11}(x) =& x^3 - 96 \cdot 11x + 112 \cdot 11^2, \\ f_{19}(x) =& x^3 - 8 \cdot 19x + 2 \cdot 19^2, \\ f_{43}(x) =& x^3 - 80 \cdot 43x + 42 \cdot 43^2, \\ f_{67}(x) =& x^3 - 440 \cdot 67x + 434 \cdot 67^2, \\ f_{163}(x) =& x^3 - 80 \cdot 23 \cdot 29 \cdot 163x + 14 \cdot 11 \cdot 19 \cdot 127 \cdot 163^2. \end{aligned}$$

Determining x mod p^2 with $p = x^2 + y^2$ and $4 \mid x - 1$

Theorem (Z. W. Sun, 2011). Let $p \equiv 1 \pmod 4$ be a prime. Write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$(-1)^{(p-1)/4} x \equiv \sum_{k=0}^{(p-1)/2} \frac{k+1}{8^k} \binom{2k}{k}^2$$
$$\equiv \sum_{k=0}^{(p-1)/2} \frac{2k+1}{(-16)^k} \binom{2k}{k}^2 \pmod{p^2}.$$

Remark. Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Z. H. Sun [Proc. AMS] proved the author's conjectural congruences

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{8^k} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p-1)/4} \left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

On x mod p^2 with $p = x^2 + 3y^2$ or $p = x^2 + 7y^2$

Conjecture (Z. W. Sun) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k+1}}{48^k} \equiv 0 \pmod{p^2}.$$

If $p \equiv 1 \pmod 3$ and $p = x^2 + 3y^2$ with $x \equiv 1 \pmod 3$, then

$$x \equiv \sum_{k=0}^{p-1} \frac{k+1}{48^k} \binom{2k}{k} \binom{4k}{2k} \pmod{p^2}.$$

Conjecture (Z. W. Sun) Let $p > 3$ be a prime. If $(\frac{p}{7}) = 1$ and $p = x^2 + 7y^2$ with $(\frac{x}{7}) = 1$, then

$$\sum_{k=0}^{p-1} \frac{k+8}{63^k} \binom{2k}{k} \binom{4k}{2k} \equiv 8 \left(\frac{p}{3}\right) x \pmod{p^2}.$$

My problems for x^2 mod p^2 with $4p = x^2 + dy^2$

Problem 1. Given a squarefree positive integer d, find *integers* $a_0, a_1, a_2, \ldots$ such that for any prime $p > 3$ not dividing d we have

$$\sum_{k=0}^{p-1} a_k \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \text{ (and } 4 \nmid x \text{ if } d = 1), \\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1. \end{cases}$$

If one thinks that the integral condition of $a_0, a_1, a_2, \ldots$ in Problem 1 is too harsh, we may study the following easier problem.

Problem 2. Given a squarefree positive integer d, find *rational numbers* $a_0, a_1, a_2, \ldots$ with denominators not divisible by large primes such that for large primes p we have

$$\sum_{k=0}^{p-1} a_k \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \text{ (and } 4 \nmid x \text{ if } d = 1), \\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1. \end{cases}$$

We find that Problems 1 and 2 have affirmative answers for most of those $d \in \mathbb{Z}^+$ with the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ having class number 1 or 2 or 4.

Apéry numbers

In his proof of the irrationality of $\zeta(3)$, Apéry introduced

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \ (n = 0, 1, 2, \ldots).$$

Conjecture (Z. W. Sun, 2010). For any odd prime p, we have

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}; \end{cases}$$

also,

$$\sum_{k=0}^{p-1} (-1)^k A_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Remark. In 2011 I proved the mod p version of both congruences and that

$$\sum_{k=0}^{p-1} (-1)^k A_k \equiv 0 \pmod{p^2} \ \text{ for any prime } p \equiv 2 \pmod{3}.$$

Solution to Problem 1 for $d = 1$

Define Apéry polynomials by

$$A_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k \ (n = 0, 1, 2, \ldots).$$

Theorem 1 (Z. W. Sun, 2011) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} (-1)^k A_k(-2) \equiv \sum_{k=0}^{p-1} (-1)^k A_k\left(\frac{1}{4}\right)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + y^2 \ (2 \nmid x), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Remark. A lemma states that for any odd prime p we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-8)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + y^2 \ (2 \nmid x), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

This was first conjectured by the author in 2009 and later confirmed by his twin brother Z.-H. Sun in 2010.

Apéry polynomials

Theorem (Z. W. Sun, 2011). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}(-1)^k A_k(x) \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}x^k \pmod{p^2}.$$

Also, for any p-adic integer $x \not\equiv 0 \pmod{p}$ we have

$$\sum_{k=0}^{p-1}A_k(x) \equiv \left(\frac{x}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p}.$$

A Key Lemma (Z. W. Sun, 2011). If x is a p-adic integer with $x \equiv 2k \pmod{p}$ where $k \in \{0, \ldots, (p-1)/2\}$, then we have

$$\sum_{r=0}^{p-1}(-1)^r\binom{x}{r}^2 \equiv (-1)^k\binom{x}{k} \pmod{p^2}.$$

More on Apéry numbers and polynomials

Conjecture (Z. W. Sun). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}(-1)^k A_k(16) \equiv \sum_{k=0}^{p-1}\binom{2k}{k}^3$$

$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1 \ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Remark. Recently Z. H. Sun confirmed the second congruence in the case $(\frac{p}{7})=-1$ via Legendre polynomials.

Conjecture (Z. W. Sun). Let $p>3$ be a prime. If $p\equiv 1 \pmod 3$, then

$$\sum_{k=0}^{p-1}(-1)^k A_k \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k} \pmod{p^3}.$$

If $p\equiv 1,3 \pmod 8$, then

$$\sum_{k=0}^{p-1}A_k \equiv \sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{256^k} \pmod{p^3}.$$

Arithmetic means involving Apéry numbers

Theorem. Let $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$.

(i) (Z. W. Sun, 2010) We have

$$\sum_{k=0}^{n-1}(2k+1)A_k \equiv 0 \pmod{n}.$$

(ii) (Conjectured by Z. W. Sun in 2010 and proved by V.J.W. Guo and J. Zeng in 2011)

$$\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k \equiv 0 \pmod{n}.$$

Richard Penner (June 2011) pointed out an application of my proof of (i):

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k = \text{the trace of the inverse of } nH_n,$$

where H_n refers to the Hilbert matrix $(\frac{1}{i+j-1})_{1\leqslant i,j\leqslant n}$.

Problem 1 for $d = 15$

For $k = 0, 1, 2, \ldots$ let $T_k = T_k(1,1) = [x^k](x^2 + x + 1)^k$.
Conjecture (Z. W. Sun, 2011). For any prime $p > 3$, we have

$$\sum_{k=0}^{p-1} (-1)^k \binom{2k}{k}^2 T_k$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \ \&\ p = x^2 + 15y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \ \&\ p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{15}) = -1. \end{cases}$$

and

$$\sum_{k=0}^{p-1} (105k+44)(-1)^k \binom{2k}{k}^2 T_k \equiv p\left(20 + 24\left(\frac{p}{3}\right)\left(2 - 3^{p-1}\right)\right) \pmod{p^3}.$$

Also,

$$\frac{1}{2n\binom{2n}{n}} \sum_{k=0}^{n-1} (-1)^{n-1-k} (105k+44) \binom{2k}{k}^2 T_k \in \mathbb{Z}^+ \quad \text{for all } n = 1, 2, \ldots.$$

Problem 1 for $d = 5$

Define polynomials

$$S_n(x) := \sum_{k=0}^{n} \binom{n}{k}^4 x^k \quad (n = 0, 1, 2, \ldots).$$

Conjecture (Sun) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} S_k(-4) \equiv \sum_{k=0}^{p-1} S_k(-64)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \ \& \ p = x^2 + 5y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \ \& \ 2p = x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-5}{p}) = -1. \end{cases}$$

$$\sum_{k=0}^{p-1} (8k+7) S_k(-64) \equiv p \left(\frac{p}{15}\right) \left(3 + 4\left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

$$\frac{1}{n} \sum_{k=0}^{n-1} (8k+7) S_k(-64) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots.$$

Problem 1 for $d = 6$

Conjecture (Sun). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} S_k(4)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \ \&\ p = x^2 + 6y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \ \&\ p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-6}{p}) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1} (24k + 17) S_k(4) \equiv p\left(5 + 12\left(\frac{2}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1} (24k + 17) S_k(4) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots.$$

Problem 1 for $d = 30$

Conjecture (Sun) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} S_k(36)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{p}{3}) = (\frac{p}{5}) = 1,\ p = x^2 + 30y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{3}) = 1,\ (\frac{2}{p}) = (\frac{p}{5}) = -1,\ p = 3x^2 + 10y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = 1,\ (\frac{p}{3}) = (\frac{p}{5}) = -1,\ p = 2x^2 + 15y^2, \\ 2p - 20x^2 \pmod{p^2} & \text{if } (\frac{p}{5}) = 1,\ (\frac{2}{p}) = (\frac{p}{3}) = -1,\ p = 5x^2 + 6y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-30}{p}) = -1. \end{cases}$$

$$\sum_{k=0}^{p-1} (8k+7) S_k(36) \equiv p\left(\frac{p}{15}\right)\left(3 + 4\left(\frac{-6}{p}\right)\right) \pmod{p^2}.$$

We also have

$$\frac{1}{n}\sum_{k=0}^{n-1} (8k+7) S_k(36) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots.$$

Problem 1 for $d = 190$

Conjecture (Sun) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} S_k(5776)$$

$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if}, (\frac{2}{p}) = (\frac{p}{5}) = (\frac{p}{19}) = 1, p = x^2+190y^2, \\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p}) = 1, (\frac{p}{5}) = (\frac{p}{19}) = -1, p = 2x^2+95y^2, \\ 2p-20x^2 \pmod{p^2} & \text{if } (\frac{p}{19}) = 1, (\frac{2}{p}) = (\frac{p}{5}) = -1, p = 5x^2+38y^2, \\ 2p-40x^2 \pmod{p^2} & \text{if } (\frac{p}{5}) = 1, (\frac{2}{p}) = (\frac{p}{19}) = -1, p = 10x^2+19y^2 \\ 0 \pmod{p^2} & \text{if } (\frac{-190}{p}) = -1. \end{cases}$$

$$\sum_{k=0}^{p-1}(816k+769)S_k(5776) \equiv p\left(\frac{p}{95}\right)\left(361+408\left(\frac{p}{19}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(816k+769)S_k(5776) \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \ldots.$$

Problem 2 for $d = 11, 35$

Conjecture (Sun, 2010). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}$$

$$\equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{11}) = 1 \ \&\ 4p = x^2 + 11y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11}) = -1, \text{ i.e., } p \equiv 2, 6, 7, 8, 10 \pmod{11}. \end{cases}$$

$$\frac{1}{p}\sum_{k=0}^{p-1} \frac{11k+3}{64^k}\binom{2k}{k}^2\binom{3k}{k} \equiv 3 + \frac{7}{2}p^3 B_{p-3} \pmod{p^4}.$$

Conjecture (Sun, 2011). Let $p \neq 2, 7$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{(-28)^k} \sum_{j=0}^{k} \binom{k}{j}^2 \binom{k+j}{j}$$

$$\equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{5}) = (\frac{p}{7}) = 1 \ \&\ 4p = x^2 + 35y^2, \\ 2p - 5x^2 \pmod{p^2} & \text{if } (\frac{p}{5}) = (\frac{p}{7}) = -1 \ \&\ 4p = 5x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{35}) = -1. \end{cases}$$

Philosophy about Series for $1/\pi$

I formulated the following viewpoint the initial version of which appeared in my message to Number Theory Mailing List sent on March 30, 2010.

Philosophy about Series for $1/\pi$. Given a *regular* identity of the form

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi},$$

where $a_k, b, c, m \in \mathbb{Z}$, bm is nonzero and C^2 is rational, there exist an integer m' and a squarefree positive integer d with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1, 2, 2^2, 2^3, \ldots\}$ (and with $C/\sqrt{d}$ often rational) such that either $d > 1$ and for any prime $p > 3$ not dividing dm we have

$$\sum_{k=0}^{p-1}\frac{a_k}{m^k} \equiv \begin{cases} (\frac{m'}{p})(x^2-2p) \pmod{p^2} & \text{if } 4p = x^2+dy^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1, \end{cases}$$

or $d = 1$, $\gcd(15, m) > 1$, and for any prime $p \equiv 3 \pmod 4$ with $p \nmid 3m$ we have $\sum_{k=0}^{p-1} a_k/m^k \equiv 0 \pmod{p^2}$.

Illustrating the Philosophy by an Example

Recall my following conjectural series

$$\sum_{k=0}^{\infty}\frac{1638k+277}{(-240)^{3k}}\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)=\frac{44\sqrt{105}}{\pi}.$$

Actually this identity was motivated by the following conjecture.

Conjecture (Sun). Let $p>5$ be a prime. Then

$$\left(\frac{15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)}{(-240)^{3k}}$$

$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{13})=1\ \&\ 4p=x^2+91y^2,\\ 2p-7x^2 \pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{13})=-1\ \&\ 4p=7x^2+13y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{91})=-1.\end{cases}$$

$$\sum_{k=0}^{p-1}(1638k+277)\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)}{(-240)^{3k}}$$

$$\equiv\frac{p}{40}\left(8701\left(\frac{\mathbf{-105}}{p}\right)+2379\left(\frac{735}{p}\right)\right) \pmod{p^2}.$$

Another Example Illustrating the Philosophy

Recall my following conjectural series

$$\sum_{k=0}^{\infty}\frac{80k+13}{(-168^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)=\frac{14\sqrt{210}+21\sqrt{42}}{8\pi}.$$

Actually this identity was motivated by the following conjecture.

Conjecture (Sun). Let $p>3$ be a prime with $p\neq 7$. Then

$$\left(\frac{-42}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)}{(-168^2)^k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15}\ \&\ p=x^2+15y^2,\\ 12x^2-2p \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15}\ \&\ p=3x^2+5y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{15})=-1.\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{80k+13}{(-168^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)$$

$$\equiv p\left(3\left(\frac{\mathbf{-42}}{p}\right)+10\left(\frac{\mathbf{-210}}{p}\right)\right) \pmod{p^2}.$$

The Third Example Illustrating the Philosophy

Conjecture (Z. W. Sun, 2011). Let $p > 3$ be a prime. Then

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{256^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k} 12^{n-k}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1,7 \pmod{24} \ \&\ p = x^2 + 6y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5,11 \pmod{24} \ \&\ p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-6}{p}) = -1. \end{cases}$$

And

$$\sum_{n=0}^{p-1} \frac{6n-1}{256^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k} 12^{n-k} \equiv -\mathbf{p} \pmod{p^2}.$$

Also,

$$\sum_{n=0}^{\infty} \frac{6n-1}{256^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k} 12^{n-k} = \frac{8\sqrt{\mathbf{3}}}{\pi}.$$

One More Example Illustrating the Philosophy

Conjecture (Z. W. Sun, 2011). For $n \in \mathbb{N}$ define

$$P_n^+(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} x^{2k-n}.$$

Then

$$\sum_{k=0}^{\infty} \frac{8851815k + 1356374}{(-29584)^k} \binom{2k}{k} P_k^+(175) = \frac{1349770\sqrt{7}}{\pi}.$$

Also, for any odd prime $p \neq 43$ we have

$$\sum_{k=0}^{p-1} \frac{8851815k + 1356374}{(-29584)^k} \binom{2k}{k} P_k^+(175)$$
$$\equiv p \left(1300495 \left(\frac{p}{7}\right) + 55879\right) \pmod{p^2},$$

One More Example Illustrating the Philosophy (continued)

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} P_k^+(175)}{(-29584)^k}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{-1}{p}) = (\frac{p}{7}) = (\frac{p}{19}) = 1, \\ & \text{and } p = x^2 + 133y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{7}) = 1, (\frac{-1}{p}) = (\frac{p}{19}) = -1, \\ & \text{and } 2p = x^2 + 133y^2, \\ 2p - 28x^2 \pmod{p^2} & \text{if } (\frac{p}{19}) = 1, (\frac{-1}{p}) = (\frac{p}{7}) = -1, \\ & \text{and } p = 7x^2 + 19y^2, \\ 2p - 14x^2 \pmod{p^2} & \text{if } (\frac{-1}{p}) = 1, (\frac{p}{7}) = (\frac{p}{19}) = -1, \\ & \text{and } 2p = 7x^2 + 19y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-133}{p}) = -1. \end{cases}$$

Remark. The class number of $\mathbb{Q}(\sqrt{-133})$ is four.

My secret criterion for existence of series for $1/\pi$ of type IV

Hypothesis (Sun, 2011). (i) Suppose that

$$\sum_{k=0}^{\infty}\frac{a_0+a_1k}{m^k}\binom{2k}{k}^2T_{2k}(b,c)=\frac{C}{\pi}$$

with $a_0,a_1,b,c,m\in\mathbb{Z}$, $b,c>0$ and $C^2\in\mathbb{Q}\setminus\{0\}$, and that $p^2\nmid c$ for any prime $p\mid b$. Then $c=1$, and $\sqrt{|m|}$ is an integer dividing $16(b^2-4)$. Also, $b=7$ or $b\equiv 2\pmod 4$.

(ii) Let $\varepsilon\in\{\pm1\}$, $b,m\in\mathbb{Z}^+$ and $m\mid 16(b^2-4)$. Then, there are $a_0,a_1\in\mathbb{Z}$ such that

$$\sum_{k=0}^{\infty}\frac{a_0+a_1k}{(\varepsilon m^2)^k}\binom{2k}{k}^2T_{2k}(b,1)=\frac{C}{\pi}$$

for some $C\neq 0$ with C^2 rational, if and only if $m>4(b+2)$ and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(b,1)}{(\varepsilon m^2)^k}\equiv\left(\frac{\varepsilon(b^2-4)}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(b,1)}{(\varepsilon\bar{m}^2)^k}\pmod{p^2}$$

for all odd primes $p\nmid b^2-4$, where $\bar{m}=16(b^2-4)/m$.

Summary

Problem 1 for $d = 1$ already has a positive answer.

We suggest positive answers to Problem 1 for

$$d \in \{2, 3, 5, 6, 7, 10, 13, 15, 22, 30, 37, 58, 70, 85, 130, 190\}.$$

We also formulate many conjectures concerning Problem 2; in particular, we give explicit conjectural positive answers for those squarefree positive integers d with $\mathbb{Q}(\sqrt{-d})$ having class number at most two except for $d = 187, 403$.

Note that $\mathbb{Q}(\sqrt{-d})$ has class number two if and only if

$$d \in \{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, \\ 91, 115, 123, 187, 235, 267, 403, 427\}.$$

Connections of Problems 1 and 2 to series for $1/\pi$ are very mysterious!
Many conjectures of mine might remain open for many years!
For more detailed survey, the reader may consult my preprint available from `http://arxiv.org/abs/1103.4325`

Thank you!